# AN INVESTIGATION INTO THE NATURE OF MATHEMATICAL MEANING

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## 1. Introduction

Recently, concepts as mathematical understanding and mathematical meaning are becoming issues in philosophy of mathematics. Particularly interesting is that such topics border on semantics. one earnestly wonder how semantical issues can be sorted out in mathematics even though any mention of number calls into the human mind images of its supposed meaning and value. We may even be able to claim this knowledge but that would be as simple intuition which is readily crippled as we try to formalize it or as our numbers increase. Perhaps the difficulty of any industry aimed at deducing the semantics of mathematics can be captured by what I shall like to call the problem of mathsemantics as stated bellow:

1ne + 3ree is 2rue,
If it is 2rue then it is 4alse,
If it is 4alse then it is 2rue,
It can't be neither 2rue nor 4alse,
So it is either 2rue or 4alse,
But which is it?

Problem of mathsemantics is a quest to ascertain given any numbers in mathematics whether they have fixed meanings or just empty symbols connoting different things at different times.

In logical terms, I do not think there is any known algorithm that can solve it. In fact, this problem portends a kind of set never before considered i.e. a non-empty set of empty sets. We talk of an empty set as a set with no members but not of a set whose members are sets and yet this set is empty. This may look puzzling for how can a thing be and not be at the same time. The problem of mathsemantics creates this possibility in that in a non-empty set of empty sets, the member sets are not elements or





objects of mathematical accretions but metamathematical properties which cannot be formalized within set theoretic terms. We know that adding 1+ 3 cannot yield 2 but it can yield a result '4' that is true even if not provable, it is intuitively given. If the property 2rue is an element of the set 1ne + 3ree then the property 4alse is also an element of the set 1ne + 3ree. But the metamathematical properties true and false cannot both be elements of same set else the set would be saying of itself “it is not the case that I exist” which sounds like self-contradictory.

Premise 4 tells us that the set necessarily has members. The conclusion says it is one of the two properties but not both. But this is not decisive hence a question arises: which one of the two? This becomes dicey because premises 2 and 3 say that the presence of either of the two implies the other thus we see that this set of sets has members and at the same time is empty. This is different from the popular empty set which is supposed to have members but turned out not to have any. The non-empty set of empty sets which the problem of mathsemantics portends actually has members in mathematical terms but in metamathematical terms is empty. Now, this is not exactly a clear case of self-contradiction if we decide that mathematics and metamathematics are not in one-to- one correspondence. But we cannot avoid stating them as a simple case of logical consequence. Take modus ponens as an example p, p  q, q. If we assert p and then q fails to pop up, how do we solve the problem? This problem of mathsemantics which is concerned with establishing the meaning of number has yielded a non-empty set of empty sets –this I shall dub “the axiom of failed set!” It is a failed set which has members and at the same time no members or says of itself, “I have members who do not have any members”. It is a failed set also because a set is supposed to have collections, if it does not then it is empty but when its collections are said to be there and at the same time (colloquially) cannot be found; then it is not just empty but suffers a failure. The problem now is, how can we prove that this failed set has elements (is not empty) or has no elements (is empty)?

Axiom of the failed set:
There exists a set whose members are sets with no members.
$ x $y "z (z Î y = Æ)  É  (y  Î x ¹ ¢)

We cannot use the emptiness sign Æ for it is neither proven empty nor nonempty, so we introduce the sign ¢ to indicate this. Interestingly the failed set has little resemblance to the set of sets which generates the

Russell paradox i.e. axiom of comprehension. We can construct two sets (a,b) and (b,c) and then the third which would have just those two sets as members {(a, b), (b, c)}. Now one may ask: is this third set a member of itself? Axiom of comprehension shows that it is. We can call it "R":

R = {X| X ÏX}.
But is R not a member of itself?
RÎ R iff R Ï R.

This generates a self-contradiction for if R is a member of itself then it is not, and if it is not then it is.

However, axiom of failed set surprisingly does not generate a self-contradiction like the one above. Let us construct two empty sets ( )and ( ), and then the third which would have just those two empty sets as members {( ), ( )}. This third set would be nonempty because it has members, so let us call it R' and the two empty sets we may call R.

R'¹ {X|X ÎÆ}
First we ask: Is R' a member of itself?
R' Ï R' iff R Î Æ
R Î Æ
R' Ï R'.

Here, we can see no contradiction or Russell paradox arises. R' is not identical with R, thus, R' is not a member of R' if and only if R is an empty set. R is shown to be an empty set so R' is not a member of R'. Hence R' is not a member of itself so the axiom of failed set is consistent however its consistency ironically yields a problem in another aspect of mathematics.

This astonishing result is actually a problem in disguise when we shift base to mathematical meaning or mathsemantics. Let us again replicate our sets: two empty sets ( ), ( ) and a third presumed non-empty because it contains those two as members {( ), ( )}, let us call the third R' and the other two R as before.
Second we ask: Is R' empty or non-empty set?
Case I: R' is non-empty because it has members.
R' = {X| X Î R'}
Case II: R' is empty because its members R is empty.
R' ¹ {X|X Î Æ É X Ï R'}

Case one suggests that since R are sets and are members of R', R' is not empty even if R is. Simple rules of sets and membership cover that. But case II is as strong as case I, it says if R is empty and is the member of R'





then by default, R' is empty as well. There are a host of rules that support this, hypothetical syllogism, induction and compositionality principles to mention a few. Hence, is R' a non-empty or empty set?

A failed set is a power set whose members are empty subsets. So a failed set is not necessarily empty, it is its members that are empty. But it can be argued that since its members are empty, it is empty as shown above. Again failed set has no proper subset and which is worse, the union, intersection and complement of its subsets are all identical. i.e.Æ. Hence, two subsets of a failed set cannot form a power set. E.g.

A = Æ
B = Æ
A U B = Æ

It is only if a failed set has a proper subset that it can be a power set. Since it does not, it shows that it is not necessarily a power set as it appears. Again, there are no necessary and sufficient conditions to show that it is a power set.
AUB is a power set:

Iff A Ì B, but A Í B is the case so A U B might as well be either A or B. And there is no way to prove that A U B is actually A U B so the subsets in a failed set cannot form a power set properly so-called. Thus,
P (A U B) É (A B) É (A - B) = A v B.
And any of A or B is not a power set. Therefore, by excluded middle, if the set A is empty, it is not the case that it is non-empty. So the presumed non-empty power set of a failed set is once again reduced to an empty single set. Therefore a failed set which is self evidently non-empty is here shown to be self evidently empty. I for one, think this problem of mathsemantics has no solution in so-far-as- we arrogate content in our conception or definition of number, but what implications would it have for the philosophy of mathematics? The main implication has to do with the nature of mathematical meaning. To obtain the accurate insight we suggest in this work a revisit to the concept of number and by extension all mathematical objects. The theory we are projecting to cushion the limitations of the post Fregean schools like logicism, formalism, intuitionism etc., is called functionalism. It primarily conceives mathematical objects as symbols without contents whose meanings are functionally derived in application to empirical things. Based on this new foundation, functionalism revisits the main questions of philosophy of

mathematics namely: nature of mathematical truth, mathematical meaning as well as mathematical understanding. Let us first of all look for the meaning of mathematical objects across the lawn of history. Already, philosophers of mathematics are beginning to ask anew: given the numbers, the functors, the symbols and the signs, what do we mean when we speak of them? Just like the sciences, the social sciences, the humanities and even the religions know what they mean when they state their theories or creeds.

What then is mathematical meaning? Only a few years ago, an interesting philosophical question began to receive attention, it concerns logic and philosophy of mathematics and it claims that a "good" mathematical proof should do more than to convince us that a certain statement is true. It should also explain why the statement in question holds, that question is: what is mathematical understanding?1 I have discovered in my recent researches that we cannot "effectively" talk of "mathematical understanding" without first having talked of "mathematical meaning". Our understanding of why certain propositions that are either true or false actually hold must to some extent depend on the meaning of such propositions. It is in this light that I think "mathematical understanding" intersperses "mathematical meaning". For indeed, when we construct a mathematical proposition from atomic formulas and go ahead to draw useful proofs from them must they not be meaningful in themselves? Hence, before a "good" proof should be able to explain to us why a proposition that is true holds, the proposition and indeed, its atomic formulas must be meaningful concepts or numbers or symbols etc.

In this work, we take a new look at the question of mathematical meaning as reduced to the concept of number. Since number is the basic building block of mathematics and most mathematical propositions are constructed with numbers or symbols that represent numbers, I shall limit this enquiry to the meaning of number.

In this investigation, some of my key questions shall be: is number meaningful? What is the nature of mathematical proposition and of the concept of number? If numbers are meaningful, what is the character of their meanings and can this be proved in a formal system? In the section to follow, I shall attempt firstly, to define our concepts, for as





Frege would say (xii) when we ask someone what the number one is, or what the symbol 1 means, we get as a rule the answer "why, a thing. But is "the number one is a thing" a definition? The answer is "no" because it has a definite article on one side and an indefinite article on the other. Besides, it only assigns the number to the class of things, without stating just what that thing is. Our investigation here is to decipher just what that number 1 as a symbol means, in other words, the meaning of 1. It is also important to point out from this outset that we distinguish 'figure' and 'word' in this inquiry and as such '1' and 'one' are not conceived to be equal, by extension mathematical proposition and linguistic proposition. Therefore all mathematical objects and their extension into mathematical propositions are the focus of this work in which it is shown that the character of one-to-one correspondence of the principle of identity ranges not only over numeric property but over the semantic property as well. Perhaps Putnam's classic example will help us draw the line here: "there are just as many stars in galaxy A as in galaxy B." On the most natural reading, this means that there exists a one-to-one correspondence between the stars in galaxy A and the stars in galaxy B. Mathematically, it only means that the numbers of stars in the two galaxies are the same, it does not mean as well that the two groups of stars are the same or identical. So identity ranges over only denumerable extensions in linguistic propositions and never over semantic extensions. This becomes clear when we construct a mathematical proposition.

E.g. $2 = 2$

Here we are not only saying that the numbers of 2 and another 2 are the same (denumerable extension) but our theory of identity also supposes that 2 and another 2 are ontologically the same (semantic extension), such that when we prove this proposition we say it is true. Similarly, when we write $2 = 3$ and our proof says something other than what the proposition claims, we say it is false. So the values we assign to our mathematical propositions are not only results of their denumerable extensions but those of their semantic extensions as well. In the Putnam example cited above, our identity principle could not range over the semantic extension of the proposition because of the well known problems of equivocation associated with natural language. Such problems include context restrictions, problems of translation and transliteration, and of course that of shift. It is important to note here that

Frege's treatment of his theory of sense and reference or denotation as some translations would have it, is restricted to linguistic propositions. He never really extended this project to mathematical propositions as far as we can tell. This is in part the goal of our investigation here.

**2. Meaning of Number**

**i. Frege and Others:**

Defining number is not an easy task and I shall reserve it for last. However, popular conceptions following the referential theory of meaning have it that mathematical object as number is a place holder for characterizing things in the world. The current president of Nigeria is Goodluck Jonathan. This proposition conveys a picture of a man called Goodluck Jonathan whom it refers to. But mathematical propositions as 4 + 4 cannot be said to refer to 8, for even then, 8 would be required to refer to something else as its meaning. And so, are the numbers and by extension, mathematical propositions meaningless?

Obviously, this cannot be the case for we understand mathematics and the application of number but what do they refer to? They cannot be meaningful in themselves or they will be semantically closed. Also, numbers are not meaningful in themselves or they will not be called symbols. If they are meaningful in themselves, such meanings cannot be known. And if such can be known, they cannot be proved. In classical logic, that which cannot be proved within a given system cannot be true within it. This also makes appropriate Frege's dictum that, in mathematics, only that may be taken to exist whose existence has been proved. Hence meaning of numbers and the nature of mathematical propositions are stellar issues. Shapiro writes that to some extent, some questions concerning the applications of mathematics are among this group of issues. What can a theorem of mathematics tell us about the natural world studied in science? To what extent can we prove things about knots, bridge stability, chess endgames, and economic trends? There are (or were) philosophers who take mathematics to be no more than a meaningless game played with symbols , but everyone else holds that mathematics has some sort of meaning. What is this meaning, and how does it relate to the meaning of ordinary nonmathematical discourse? What can a theorem tell us about the physical world, about human knowability, about the abilities-in-principle of programmed computers, and so on?(9)





Let me turn to Frege who devoted much attention to the concepts of meaning and number. Frege noticed the difficulty involved in defining number. These numbers do not have objective characteristics which make them seem subjective. They are not objects which makes them seem like abstract concepts.

The problem is just how do you conceive them? logical or psychological? In his work The Foundations of Arithmetic – a Logico-Mathematical Enquiry into the Concept of Number, Frege spelt out what he called the three fundamental principles which will guide this type of enquiry. They are as follows (xxii).

1. Always to separate sharply the psychological from the logical, the subjective from the objective.
2. Never to ask for the meaning of a word in isolation, but only in the context of a proposition.
3. Never to lose sight of the distinction between concept and object.

Let me come straight to the second principle. If this principle is not observed in any enquiry regarding the meaning of words or even symbols, one would find himself taking as their meanings mental pictures or acts of the human mind just like the Wittgensteinian position cited above. In this way also, one would contravene the first principle since one would be allocating psychological meaning to logical property or taking the meaning of an objective entity to be subjective.

Frege's insistence that words have meanings only in the context of propositions where they appear readily applies to symbols. The problem is that words are less flexible and dynamic than symbols. Let us therefore add here a fourth principle:

4. Always distinguish between words and symbols.

This fourth principle is important in our enquiry because words are a collection of letters but symbols are both collection of figures and letters – these letters however cannot be words otherwise they cannot be symbols. Example: “E”, “A”, “G”, “L” and “E” are letter symbols but “Eagle” is a word, both are therefore not the same and cannot have the same meaning.

If we apply the second principle to our present investigation, it does follow that the meanings of individual symbols are not to be sought except within the context of mathematical or formal propositions. A symbol will therefore be meaningful only within a proposition otherwise, it will be … what? Meaningless? Of course, Frege did not imply this. It seems his insistence on finding meaning of words within the context of propositions is because the property of "meaning" is to him not a permanent attribute of words. In his "Sense and Meaning" he identifies meaning of words with the senses of their usage in a context of a proposition (Lindberg, 61 -74). In such propositions, what the words refer to which constitute their meanings are the given senses of their usage within a specific context. The major fallout from this is that meaning is not fixed, as a linguistic concept. If we apply it to our present investigation, it will also show that as a formal concept meaning is not fixed – it is therefore dynamic, changing from context to context of propositions. This shows that we cannot talk of a meaningful symbol (in isolation) but only the meaning of a symbol (in the context of a proposition).

Yet in mathematics as in logic, we very often use symbols in isolation of any formal proposition and such isolated symbols have fixed unbemused meanings. For a simple example I can decide to write out the logical operators as follows:

1. &
2. v
3. É
4. 
5. ~

These symbols are not in the context of any formal propositions but they have fixed unbemused meanings. Unlike the following;

1. Eagle
2. Peacock
3. Lion
4. Swine

Frege's contention pans out here, since one may not know what is meant by eagle, a high flying bird or the attribute of distinction, beauty etc., or by peacock is meant a multi-colored bird or pride; or by lion is meant a beast or courage or brevity; or by Swine is meant a mammal or dirtiness or unholy. We therefore come to the realization that Frege's second principle cannot necessarily apply to our investigation in this way which concerns neither words nor natural language but symbols and formal language. But even if it does, his context principle (no.2 above) still restricts the meaning of number to the context of its use, in which





case number 7 for example will be though denumerable but infinite according to the number of its usage, individuals and contexts, this is not feasible!

In his theory of "Sense and Reference", Frege appears to strengthen his views concerning the meaning of number beyond his famous context principle, to "never ask for the meaning of a word in isolation, but only in the context of a proposition". Here, he espouses that number can have a reference and a sense according to usage (context of proposition), for example, "4 x 2" as a mathematical expression stands for '8', but 11 – 3 is a way that the number 8 is determined or picked out. If the former is the reference (or what the number refers to) following the referential theory of meaning, the latter would be one of the sense of the mathematical expression. But Frege's enterprise fails for example; what would 4 2 or 5 + 3, or 4 + 4 or 7+ 1 = 8 be called? Is '8' the sense or reference in which one and why? There is no difference between these expression 4 + 4, 4 x 2 and 11 – 3 except that they are sum, product and difference in that order. Unlike Frege's suggestions, there is nothing fundamental that would make one determine the reference of numbers and others their sense. Hence, Frege's theory of sense and Reference cannot adequately account for the distinction he anticipates to exist in the sense and reference of number. What this theory shows concerning the meaning of number is that all mathematical expressions like the ones stated above which uniformly yield number 8 can in different ways qualify as both senses and references- so there is no distinction after all. By the principle of identity therefore, sense and reference are formally the same, and by Leibnitz's law what belongs to one belongs to the other. But if we are to incline ourselves to Frege's distinction, then the fall out is that for any given number, there would be infinitely many number of sense and references for it in which case the theory of mathematical meaning is defeated. This is because by his context principle, there are infinitely many contexts in which a given number could be used and by his sense and reference, there is no rigid rule that determines what qualifies as a sense and what qualities as a reference of a given member.

Also in the formal language of arithmetic and logic, what might be the nature of arithmetical propositions since the individual symbols are capable of being meaningful even in isolation?

If I state the following formal proposition: 2 + 2 = 4

By Frege's second principle, we may wish to know what the individual symbols 2, 2, 4 mean in the context of the proposition. In linguistic propositions we may begin by consulting a dictionary, a thesaurus, a reference book and then narrowing the results down to the context of their usage; in a formal proposition such as we have above, we talk of proof. It is therefore hoped that the proof that 2 + 2 = 4, if successful will account for the meanings of the symbols in the propositions. But the problem arises, are numerical formulas provable?

Frege distinguishes numerical formulas, such as 2+ 3 = 5, which deal with particular numbers, from general laws, which hold good for all whole numbers (5). The former are held by some philosophers to be unprovable and immediately self-evident like axioms (Hobbes, 19 and 62 – 63: Locke,2 IV, 6: Newton,3 iii, 24). Kant on his own says they are unprovable and synthetic (157) but Frege hesitates to designate them as axioms because they are not general and because the number of them is infinite. To return to one of the claims made, are the particular numbers really and in all cases self-evident? If they are, then we might begin to think of arithmetical formulas truly unprovable at least in some sense. Let us consider this:
7165 + 17928 = 25093 the above formula is certainly not self-evident. Hence, when Kant thinks we can call on our intuition of fingers and points for support, he was thinking of only small numbers so that the large numbers would be provable. Yet again, this will lead Kant into the mistake of treating these propositions as empirical as opposed to his view that they are synthetic a priori, for whatever our intuition of 7165 fingers may be, it is at least clearly not pure. On the whole, if the numerical formulas were provable from say, 100 on, we should ask with fairness, "why not from 6 on? or from 2 on? or from 1 on? How can it be that some are self-evident and others provable?

Leibniz4 is one of the very few who grant that numerical formulas are provable whether small or large. In his words:

"It is not an immediate truth that 2 and 2 are
4; provided it be granted that 4 signifies 3 and
1. It can be proved, as follows:
Definitions: (1) 2 is 1 and 1



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(2) 3 is 2 and 1
(3) 4 is 3 and 1
Axiom: if equals be substituted for equals, the
equality remains.
Proof: 2 + 2 = 2 + 1 + 1 (by Def. 1) = 3 +1 (by
Def. 2) = 4 ( by Def. 3).
2 + 2 = 4 (by the Axiom)" (363).

However, Frege points out an omission (7). If we look more closely, we can discover a gap in the proof which is concealed owing to the omission of the brackets. To be strictly accurate, that is, we should have to write:

$$2 + 2 = 2 + (1 + 1)$$
$$(2 + 1) + 1 = 3 + 1 = 4$$

What is missing here is the proposition

$$2 + (1 + 1) = (2 + 1) + 1$$

Which is a special case of

$$a + (b + c) = (a + b) + c$$

Grassmann and Henkel also share this opinion which is associativity. If we assume this law, it is easy to see that a similar proof can be given for every formula of addition. Be that as it may, the proof of 2 + 2 = 4 has yet to tell us the meanings of the symbols involved.

John Stuart Mill5 is of the opinion that mathematical symbols cannot be said to be meaningful unless they refer to observable facts. Numbers have senses which are their meaning. The sense of a number stems from the observable fact which it refers to. But there are two problems to this position. Firstly, do all numbers large and small refer to observable facts? Secondly, what would a number like 0 refer to? A 0 orange perhaps – I believe no one has ever observed that. Even if Mill argues that 0 is a number or symbol with no sense how could he defend that? A number with no sense would be an empty symbol but our calculations reveal that 0 is not empty. When for example we add 10 to 10 and obtain 20, it strikes a quick chord in us that 0 is rich with sense, for if it were not so, why did 10 + 10 not yield 2? The fact that 1 + 7 gives us 8, and when we introduce 0 in front of 1, the result increases tremendously to 17 shows that 0 could have a sense equal to 9, even though we do not know this for certain. It would be wrong for Mill to deny a sense to 0, just as it would be for anyone to assert that the observable fact which, according to Mill, is contained in the definition of an eighteen – figure number has ever been observed.

Mill also proposes to make use, for his proof of the formula 5 + 2 = 7, of the principle that "whatever that is made up of parts, is made up of parts of those parts" (5). This he holds according to Frege (12) to be an expression in more characteristic language of the principle familiar in Leibniz in the form "the sums of equals are equals". Mill calls this an inductive truth and a law of nature of the highest order. Thus, Mill understands the symbol + in such a way that it will serve to express the relation between the parts of a physical body or of a heap and the whole body or heap; but such is not a sense of that symbol. That if we heap 2 unit volumes of rice into 5 unit volumes of rice we shall have 7 unit volumes of rice, is not the meaning of the proposition 5 + 2 = 7, but an application of it, which only holds good provided that no alteration of the volume has occurred. Hence, Mill always confuses the applications that can be made of an arithmetical propositions, which often are physical and to presuppose observed facts, with the pure mathematical proposition itself.

Further still, in his analysis of the natural numbers, Frege tells us (Basic Laws, ix) that the "fundamental thought" on which his analysis of the natural or "counting numbers" is based, is the observation that a statement of number involves the predication of a concept of another concept; numerical concepts are concepts of "Second Level", which is to say, concepts under which concepts (of first level) are said to fall. This according to Demopoulos and Clark (133) yields an analysis of the notion of a numerical property, as when we predicate of the concept horse which draws the king's carriage the properties are first order definable in terms of the numerically definite quantifiers. In order to pass from the analysis of numerical properties to the numbers, Frege introduced the "cardinality principle" (Hume's principle), which he "defined" contextually as:

Nx Fx = Nx Gx  F ¡ G,

i.e., the number of Fs is the same as the number of Gs if and only if the Fs and the Gs are in one to one correspondence.

This contextual definition and the fundamental thought yield Frege's account of the applicability of mathematics. In this, the simplest case for which the question arises – the application of the cardinal numbers – the solution is that arithmetic is applicable to reality because





the concepts, under which things fall, fall under numerical concepts. For example, I have 12 cars; 12 is a numerical concept under which the concept of car falls. Thus, the numerical property 12 is here applied to the property of car but this cannot account for the meaning of the numerical symbol 12 or for the meaning of numerical or formal propositions.

To return to the nature of formal propositions, Kant says they are synthetic a priori. Immanuel Kant's thesis that arithmetic and geometry are synthetic a priori was a heroic attempt to reconcile these features of mathematics. According to Kant, mathematics relates to the forms of ordinary perception in space and time. On this view, mathematics applies to the physical world because it concerns the ways that we perceive the physical world. Mathematics concerns the underlying structure and presuppositions of the natural sciences. This is how mathematics gets "applied." It is necessary because we cannot structure the physical world in any other way. Mathematical knowledge is a priori because we can uncover these presuppositions without any particular experience (Shapiro, 5). Kant's position makes intuition the ultimate ground of our mathematical knowledge. But we have earlier showed that large numbers upon calculations are not intuitively self-evident. This will impede upon our modest efforts to obtain the meaning of number. Frege on his part says that formal propositions are analytic. If this is so, then the predicate symbol will normally be contained in the subject symbol – this will make formal propositions semantically closed and therefore incapable of yielding the meanings of number. Furthermore, formal propositions cannot be wholly empirical as Mill supposes otherwise; we would contravene Frege's first and third principles. Shapiro notes that the conflict between rationalism and empiricism reflects some tension in the traditional views concerning mathematics, if not logic. Mathematics seems necessary and a priori, and yet it has something to do with the physical world. How is this possible? How can we learn something important about the physical world by a priori reflection in our comfortable armchairs? (5). Indeed, the true nature of formal or numerical or mathematical propositions remains an open question for philosophers of mathematics.

Let us then return to the question about the nature of number or symbol contained in the mathematical propositions themselves. Readily, the question which comes to mind is how might number be defined? A definition of number possibly represents a window into its meaning according to the assumption that definitions of concepts or objects ipso fact contain ideas of their meaning. But in the case of number, no reputable philosopher I know has been bold to offer a helpful definition. In fact, Hankel6 is of the opinion that number is not definable. In his words, “what we mean by thinking or putting a thing once, twice, three times, and so on, cannot be defined, because of the simplicity in principle of the concept of putting”. What this means is that “putting” is a defining concept and is in the words of Russell atomic such that it cannot be further broken down. But this is not the only problem, Frege says, what about the words once, twice, three times (27), we might never be able to define them. On his part Leibniz views number as an adequate idea. What he means is that the concept of number is clear upon contact and so is every idea contained there in. this leaves open the point that number is indefinable but this could be an assumption simply because no one has succeeded in this attempt and not that it is a proven fact. I myself shall not attempt to define number at least not at this stage of the investigation. Be that as it may, I concede that we have again lost an important window which might lead us into understanding the meaning of number. But one issue that still resonates to this day is whether mathematical objects—numbers, points, functions, sets—exist and, if they do, whether they are independent of the mathematician, her mind, her language, and so on. From here onwards realists and anti-realists engage in a face off on whether mathematical objects are objective or not. Some notable realists include (Gödel [1944, 1964], Crispin Wright [1983], Penelope Maddy [1990], Michael Resnik [1997], Shapiro [1997]); anti-realists include (Michael Dummett [1973,1977]);(Geoffrey Hellman[1989] , and Charles Chihara [1990]. Other groups include the three principal schools, logicism, formalism and intuitionism as well as psychologism, factionalism, Platonism and conventionalism.

Platonists maintain that mathematical objects are abstract entities, existing outside space-time, and independent of our conceptions. But if mathematical objects like numbers are outside space-time, how do we come to know them? This leads to Benacarraf-Putnam problem of access (30-33). How are we supposed to have epistemological access to entities existing outside of our sphere? Intuitionism, the school which attributes existence of numbers to human thought claims to have answers to this. Mathematical object for them





exist as an entity in the intuitionist's thought. Heyting says that mathematical objects are by their very nature dependent on human thought. Their existence is guaranteed only insofar as they can be determined by thought (53). However, intuitionist's position is inches away from that of the psychologist who conceives mathematical objects as ideas or notions of the mind (Locke, 1). If the intuitionist is unable to account for the self-evidence of an eighteen digit number, the psychologist would be unable to defend his thesis which turns mathematics into psychology.

On the other hand, the formalist and the conventionalist deny Platonism. While the formalist favors mathematics with a more visible subject matter as forms of symbols on paper (Brouwer, 78: Von Neumann,63), the conventionalist maintain that mathematics has no objects, or if it does, they simply have the properties we assign to them by convention (Quine, 329-331). But how can the marks on the formalist's paper account for the meaning of numbers? And if mathematics has no objects, as the conventionalist insist, then it makes no sense to talk of numbers in the first place. His position is close to that of the nominalist who rejected Platonism because he found the idea of abstract object unintelligible (Benacarraf and Putnam, 23). But that the platonist's idea is unintelligible does not mean mathematical objects as numbers do not exist. The fictionalist says numbers exist but as the constructs of human imagination. This position is way out of the line as it would reduce mathematics to one of the creative arts. Logicists would not accept this, Frege for one posits that mathematical objects are objective although non-sensible. This makes logicism a little bit difficult to interpret for how can an entity be objective and at the same time non-sensible? It was Carnap (41) who made their position more explicit by capturing mathematical objects as concepts which comes to life through definition and postulates (axioms), they are therefore, objective because of their applicability to empirical world. For them therefore, a mathematical object exists if proved and would be meaningful by default. Having rejected Platonism, the one unanswered question remains: is mathematical object like number an external or internal property?

In our common usage in language, numbers function as adjectives and take such places as the words, blue, solid, heavy which have for their meanings properties of external things. But can we also think of individual numbers in this way or group them in the same class as say, color?

Scholars like Cantor7 (2 and 4) and Lipschitz (1)8 are of this bent of mind. Cantor says that mathematics is an empirical science since it considers things in the external world. Numbers for him originate only by abstraction from objects. Schroder as noted by Frege also sees number as a property of external things. He says it is modeled on actuality (21). His explanation is that number is derived from things by a process of copying the actual units with ones, which he calls the abstraction of number. The units are thus represented only in what he calls the point of their frequency. What this means is that individual numbers are derived according to the occurrence of things – thus, frequency becomes another name for number.

Baumann (669) rejects this position that numbers are properties of external things. Mill says they are physical things. Locke and Leibniz see them as existing only as notions. For Locke, they apply to men, angels, actions, thoughts – everything that either exists or can be imagined. While for Leibniz number is applicable to everything, material and non-material. In his words:

> Some things cannot be weighed, as having no force and power; some things cannot be measured, by reason of having no parts; but there is nothing which cannot be numbered. Thus number is, as it were a kind of metaphysical figure (162).

Yet a problem arises, how might we begin to decipher the meaning of that which is metaphysical? Do we suppose therefore that number is something subjective? I shall visit this later. Mill's contention that number is something physical in the sense that two pears are physically different from three pears as visible and tangible phenomena seems plausible but we cannot conclude there from that their twoness or threeness is something external or physical. Number is obviously different from an object it is attributed to. If number then, is nothing physical and probably nothing metaphysical, how else might we conceive it?





In the present context, Shapiro (10) notes that the question is whether the mathematician must stop mathematics until he has a semantics for his discourse fully worked out. Berkeley, an idealist philosopher says that number is not something fixed. As he puts it in his New Theory of Vision (109).

> It ought to be considered that number … is nothing fixed and settled, really existing in things themselves. It is entirely the creature of the mind, considering, either an idea by itself, or any combination of ideas to which it gives one name, and so makes it pass for a unit. According as the mind variously combines its ideas, the unit varies; and as the unit, so the number, which is only a collection of units, doth also vary. We call a window one, a chimney one, and yet a house, in which there are many windows, and many Chimneys, hath and equal right to be called one and many houses go to the making of one city.

In the light of this Berkeleyian insight, we can show within our context here that the meaning of number is flexible. And as Berkeley points out, many numbers may come to symbolize many things and yet all of them symbolized by a number say 1. There can be many symbols signifying different things, yet different symbols signifying the same thing. This shows that number as a mathematical symbol is dynamic and not fixed. This is probably the reason it has remained indefinable and which is worse, the reason it is hard to pin down its meaning. For when we talk about the meaning of number, we are not just talking of the meaning of number but on a larger scale, the meaning of every individual number. Since number 1 is different from number 2, and every number is not the same with any other number, it follows that our enquiry is not a trivial one. Does this therefore imply that (i) since numbers are many and different (ii) as mathematical symbols, many and different numbers can be subsumed in one or some numbers (iii) the character of being dynamic and not fixed mean that number can be accounted for with recourse to the human mind? By this I mean that number could be analyzed as a property of the mind in much the same way Kant treats it as a mental category. In this way, we can validly say that number is something subjective, but is it?

If number is something truly subjective or psychological then it cannot be realistically attributed to physical things but we agree that number is an attribute of external things. Frege holds the view that it is no more subjective than the moon is a cheese. The temptation to regard number as a subjective concept seems, according to Frege, to come from the link between the concept and reason which judges facts. In his words:

> … I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of the reasons, -for what are things independent of the reason? To answer that would be as much as to judge without judging, or to wash the fur without wetting it (36).

In this way, Frege disagrees with Schloemilch (1) who calls number the idea of the position of an item in a series. If number were an idea Frege objects, then, arithmetic would be psychology. But arithmetic is no more psychology than, say, astronomy is (37). Also, Frege goes on, if number were an idea then each will be private to individuals (37). We can therefore speak of my own four and your own four. In this way what one means by two or four will be different from what another means by these numbers. But we know and all agree that 2 + 2 equals 4. It will be strikingly absurd and the whole world of commerce will fall into chaos when the six billion world populations have six billion ideas of 2; and the forecasted population explosion of future generations come with even more diverse and intriguing ideas of two. Hence, number cannot be a subjective concept. It may not also be an objective property as Mill conceives but according to Frege, it can be non-sensible and objective (32). What Frege means here is that his conception of objectivity is something distinguishable from what is handleable or actual. In his words:

> The axis of the earth is objective so is the centre of mass of the solar system, but I should not call them actual in the way the earth itself is so. We often speak of equator as an imaginary line; but it would be wrong to call it an imaginary line in the dyslogistic





> sense; it is not a creature of thought, the product of a psychological process, but is only recognized or apprehended by thought. If to be recognized were to be created, then we should be able to say nothing positive about the equator for any period earlier than the date of its alleged creation (35).

What Frege is saying is that what is objective is recognizable by everyone and can be conceived and judged by reason. And number like the axis of the earth and the centre of mass of the solar system fall within this framework. However, what we have learnt so far is that number is neither something objective nor subjective but objective in a non-sensible way, this has yet to account for the question: what is the meaning of 2? What is the meaning of 3? What is the meaning of 5 and so on? According to Alberto Coffa [1991], a major item on the agenda of Western philosophy throughout the nineteenth century was to account for the (at least) apparent necessity and a priori nature of mathematics and logic, and to account for the applications of mathematics, without invoking anything like Kantian intuition. In his words, the most fruitful development on this was the "semantic tradition," running through the works of Bolzano, Frege, the early Wittgenstein, and culminating with the Vienna Circle. The main theme—or insight, if you will— was to locate the source of necessity and a priori knowledge in the use of language. Philosophers thus turned their attention to linguistic matters concerning the pursuit of mathematics. What do mathematical assertions mean? What is their logical form? What is the best semantics for mathematical language? The members of the semantic tradition developed and honed many of the tools and concepts still in use today in mathematical logic, and in Western philosophy generally. Michael Dummett calls this trend in the history of philosophy the linguistic turn.

In doing this, modern philosophers of mathematics concentrated more on the nature of mathematical object and mathematical truth which are superstructural questions at the expense of the substructural question: what is mathematical meaning or better still, what is the meaning of number? Frege was the last philosopher to seriously investigate this in his works already referred to above. He concluded following his context principle that the meaning of every individual number is to be found in the sense it refers to within a give context of a proposition. This position seems to have been wholly accepted in contemporary discussions by philosophers of mathematics or at least considered a trivial question. But it is no more trivial than the origin of species is to the evolutionist. Numbers are the basic building blocks of mathematics which is by far according to Heyting (53), an architectonic structure. Also, if mathematics is the world's most exact science, then the meaning of its basic building block cannot be trivialized.

Frege's conclusion that the sense which a number acquires in the context of a proposition is its meaning is rejected in this paper. The main reason for this being that since every individual number is capable of acquiring different senses in an infinite number of contexts, it follows that each number can have infinite number of meanings. But this is not the way numbers behave in the economics of everyday life, else accounting would be chaotic. 2 + 2 = 4 always and at all times is true simply because 2 has a fixed meaning just like 4 has its own, and these hold notwithstanding contexts. The main problem with Frege's theory of sense and reference and the context principle is that they lead to multiplication of meanings. And so we can see no conclusive response to the question what is mathematical meaning? To be modest, the question has not even arisen again since Frege, aside for a number of critical commentaries and revisits to Frege, prominent among them is Alonzo Church's four series paper on the formulation of the logic of sense and denotation9. In these papers the author attempts a fortification of Frege's views in what has come to be known thereafter as Frege-Church theory of sense and denotation. Here a distinction is drawn between the sense of a word in the context of a proposition and its denotation i.e. customary sense or meaning. Nathan Salmon in his paper 'A Problem in the Frege-Church theory of Sense and Denotation', argues that the shifts witnessed in translations into other languages and the fact that Frege's concept of Bedeutung or denotation acquires the character of tentativeness lead to a collapse of Frege and Church's argument about denotation. It should also be noted that similar problems have been identified by Benson Mates and partly also by Tyler Burge, Joseph Owens and Anthony Anderson10. It is in the light of this need to further the epistemology of foundations of mathematics that we pursue this inquiry here. However, one thing we can





pick from Frege is that numbers fall into codes. I can add 2 to 3 to get 5; 3 to 5 get 8; 2 to 8 to get 10. Does this mean that the coding system can account for the meaning of individual numbers? Is the meaning of 2 + 8 to be found in 10? or 10 – 2 to be found in 8? This enquiry occupies this paper presently.

**ii. Fibonacci Number Code and Zeckendorf Representation**

Number representation can be done in many ways but of interest to us are the Fibonacci code and the Zeckendorf representation (see Fraenkel "System of Numeration" for a fuller account of all known number code system).

Here is an investigation into representing numbers as sums of Fibonacci numbers. First, let us just use any Fibonacci number once in any of our sums to see what we get. For example:

i. 1 is a sum all on its own. So there is just one sum of Fibonacci numbers with a sum of 1.
ii. 1 + 1 = 2 but we are not allowing this as a Fibonacci sum since we have used 1 more than once.
iii. 2 is, however, a sum formed of Fibonacci numbers
iv. 1 + 2 = 3 and 3 is also a Fibonacci number so there are two sums for 3.
v. 1 + 3 = 4 is the only way to make a total of 4 using only Fibonacci numbers.
vi. 2 + 3 = 5 and again 5 is a Fibonacci number so there are two sums for 5
vii. 1 + 2 + 5 = 3 + 5 = 8, 8 is a Fibonacci number with three sums.
viii. 2 + 3 + 8 = 5 + 8 = 13. 13 is a Fibonacci number with three sums.
ix. 1 + 2 + 5 + 13 = 8 + 13 = 21. 21 is a Fibonacci number with four sums etc. Basically, Fibonacci numbers are obtained by adding a preceding one to its successor. As in above, we added 1 + 2 to get 3; 2 + 3= 5; 3 + 5 = 8; 5 + 8 = 13; 8 + 13 = 21; 13 + 21 = 34; 21 + 34 = 55 and so on. These are Fibonacci numbers and coding system. Eduardo Zeckendorf, a Belgian Doctor in his 1972 paper "Representation des Nombres …" (pp 179 – 182), developed a system commonly known today as "Zeckendorf representation" or "minimal form" as a basis

of Fibonacci arithmetic in which the Fibonacci numbers can be represented in sums, without of course repeating any Fibonacci number except in the 1 series

Let me explain this theorem with an example if we are to present the number 30 in the Fibonacci code, there are two things to note: first, 30 is not a Fibonacci number. Second, we are asked to present it in Fibonacci code which means we are going to use the sums of Fibonacci numbers. This is what is called Zeckendorf representation. Since the number we are to present is 30, we are to choose those Fibonacci numbers that are below 30. And so we have: 1, 1, 2, 3, 5, 8, 13, 21 as digit weights for such representation. Our Zeckendorf representation of the sums of Fibonacci numbers for the number 30 is as follows: 30 = 21 + 8 + 1 = 21 + 5 + 3 + 1 = 13 + 8 + 5 + 3 + 1 = 13 + 8 + 5 + 2 + 1 + 1. But among them it is possible to select one and only one representation 30 = 21 + 8 + 1, in which no consecutive Fibonacci numbers are being used.

Let us consider one practical application of the Fibonacci number representation in the conversion between miles and kilometers. According to Ron Knott11 we have approximately 8 kilometers in 5 miles. Since both of these are Fibonacci numbers then there are approximately Phi (1.618..) kilometers in 1 mile and Phi (0.618..) miles in 1 kilometer. The real figure is more like 1.6093… kilometers in 1 mile. This comes from the precise definition of 1 inch equals 2.54 centimeters exactly, and 100,000 centimeters make 1 kilometer. In the imperial system, 36 inches are 1 yard and 1760 yards are 1 mile.

Replacing each Fibonacci number by the one before it has the effect of reducing it by approximately 0.618 (phi) times (the ratio of a Fibonacci number to the one before it is nearly phi). So to convert 13 kilometers to miles, replace 13 by the previous Fibonacci number, 8, and 13 kilometers is about 8 miles. Similarly 5 kilometers is about 3 miles and 2 kilometers is about 1 mile.

Now suppose we want to convert 20 kilometers to miles where 20 is not a Fibonacci number? We can express 20 as a sum of Fibonacci numbers and convert each number separately and then add them up. Thus:

20 = 13 + 5 + 2

Using ¡ to stand for approximately equals and replacing 13 by 8,





5 by 3 and 2 by 1, we have

20 kms = 13 + 5 + 2 kilometers
¡ 8 + 3 + 1 miles
= 12 miles

To convert miles to kilometers, we write the number of miles as a sum of Fibonacci numbers and then replace each by the next larger Fibonacci number:

20 kms = 13 + 5 + 2 kilometers
¡ 21 + 8 + 3 miles
= 32 kilometers

There is no need to use the Fibonacci representation of a number, which uses the fewest Fibonacci numbers, but you can use any combination of numbers that add to the number you are converting. For instance, 40 kilometers is 2 x 20 miles. So 40kms is 2 x 12 = 24 miles approximately.

One might wonder what role the Fibonacci number code has to play in deciphering the meaning of number, but that is obvious. The Fibonacci representation shows if nothing else that there is something in a number beyond what it adds up to. How else can we explain the fact that 8 kilometers of Fibonacci number when converted yields 5 miles of yet another Fibonacci number? Why not 6 or 7 or 4 perhaps, but 5? Also, why are non- Fibonacci whole numbers capable of being presented in Zeckendorf sums of Fibonacci numbers? To be honest, I do not know why, but what presents itself from this is that there must be peculiar meaning resident in each number which leads to such harmony when they interplay like the examples we have shown above.

However, the fact that the sums of 1 + 2 + 5 = 8 = 3 + 5 = 8 also reveals the flexible character inherent in the Fibonacci representation. If individual numbers have meanings, it shows that such meanings are not fixed but tentative. They obviously shift from context to context thereby corresponding to Frege's rule 2 that meaning of words are to be obtained in the context of propositions where they appear. Hence, the Fibonacci representation shows that there is something in a number but whatever it is, “meaning” or “sense” such is not obvious and it is at the same time tentative. So our quest to understand the meaning of number is not solved here.

**iii. My Zero Number Code System**

It has been debated over time whether 0 is a number in the same sense as say 1 or 2 or 1,000. While some agree others disagree. It is partly for the interest of those who deny numeric capacity to 0 and partly for the interest of our investigation that I have chosen to conduct this brief inquiry.

0 is a number with full numeric capacity. However, there is a little difference between 0 and the rest. It is in addition, a building number. By this I mean it is a number capable of enriching the senses of other numbers as well as playing explanatory role in its special appearances. In an enrichment role, it appears in front of other numbers while in an explanatory or special role, it appears in the middle and behind other numbers. Hence the following theorems:

1. The numeric value of 0 is approximately 9 not exactly in the sense of 9 but in the sense of 1, 1, 1, 1, 1, 1, 1, 1, and 1. But in some ways, the former and the latter have one to one correspondence. Following this we can deduce that 1, 1, 1, 1, 1, 1, 1, 1, and 1 and 9 have the same values without having to state they are the same. My claim that the value of 0 is approximately 9 stems from the simple observation that when we place 0 in front of I, it becomes 10 and 10 minus 1 is 9.
2. Theorem "2" states that when we place 0 in front of a number (a unit) it grows as a sum of 9 in accordance with the unit in front of which it is placed. In this way 0 placed in front of 1 will grow as a sum of 9 into1 times plus 1 which will yield 10; in front of 2, it will grow as a sum of 9 into 2 times plus 2 which will yield 20 etc. Under this capacity, 0 plays enrichment roles.
3. Theorem "3" states that when two 0s are placed in front of a number, their values will be calculated differently. While the 0 closest to the number is calculated as the sum of 9 into the number of that number, the last 0 will be calculated as a product of 9 and the rest of that number + the number. For example, 400, from the sum of 94 + 4 = 40 and the product of 9 x 40 + 40, we get 400. This is an enrichment role for zero.
4. Theorem "4" states that when three 0s are placed in front of a number, we calculate the value of the last 0 as a product of 9 and the rest of the number. Example, 3000 will yield 9 x 300 + 300 which is 3000. This shows another enrichment role of zero.





5. Theorem "5" states that when n 0s are placed in front of a number, we calculate the value of the last 0 as a product of 9 and the rest of the number. Example, 4n will yield 9 x 4 "n" + 4 "n" which is 40n. This is also an enrichment role for zero.
6. Theorem "6" states that when 0 is placed behind a number it accounts for units which grow in sums example, 01 which can grow in sums till 099. This is an explanatory role for zero.
7. Theorem "7" states that when 0 is placed in the middle of numbers it indicates the silent number range from hundred, thousand, million etc., example, 1045. Here 0 indicates the silent number range of hundreds. This is another explanatory role for zero.

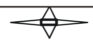
On the whole, the essence of this inquiry for our investigation is that the number zero shows that numbers possibly have values and refer to something other than them. This has the potentiality to reveal the meaning in numbers. But the question arises: is 9 the meaning of 0? or does 0 mean 1, 1, 1, 1, 1, 1, 1, 1, and 1? If we say yes then what do we understand by 9 or 1, 1, 1, 1, 1, 1, 1, 1, and 1 other than number? In simple terms: what do they mean as symbols? Certainly, our brief inquiry here has shown that there is something else behind number but whatever it is, is not obvious. So our question whether number has meaning is not fully answered by the zero-number code system. A wide call one can make from here is that mathematical objects like number may not exist after all otherwise why is it difficult to trap the meaning of numbers? The nominalist argues that there are no numbers, points, functions, sets, and so on. The burden on advocates of such views is to make sense of mathematics and its applications without assuming a mathematical ontology. This is indicated in the title of Burgess and Rosen's study of nominalism, A Subject with No Object [1997].

A variation on this theme that played an important role in the history of our subject is formalism. An extreme version of this view, which is sometimes called "game formalism," holds that the essence of mathematics is the following of meaningless rules. Mathematics is likened to the play of a game like chess, where characters written on paper play the role of pieces to be moved. All that matters to the pursuit of mathematics is that the rules have been followed correctly. As far as the philosophical perspective is concerned, the formulas may as well be meaningless. Opponents of game formalism claim that mathematics is inherently informal and perhaps even non-mechanical. Mathematical objects have meaning, and it is a gross distortion to attempt to ignore this. But I want to remain optimistic and in the following section shall consider the Igbo numeric system.

**3. Igbo Numeric System**

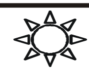
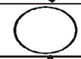
I have found among the Igbo of Eastern Niger river a curious number code system in which each of the t??ala or basic numbers has if you wish a meaning. The Igbo numeric system is divided into two: the t??ala or basic and the jik?? or building numbers. The t??ala numbers are those numbers out of which other numbers are formed and they are eleven in number namely: ı, ±, ¨, ©, ©,© , ©,©, ©, I, ıI, which immediately translates into the following Arabic numerals: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 10, the jik?? numbers on the other hand are all the numbers formed by combining the t??ala numbers, and these ranges from ıI to infinity.

However, for the Igbos, there are only eleven numbers in existence and these are the t??ala numbers, every other number is derived from this set as for example ©© is derived from © and©. In this way also, only the t??ala numbers have fixed meanings, while the jik?? get their meanings by means of derivation and permutation. When a given jik?? number is derived, the meanings of the individual t??ala numbers which consisted it will be permuted to obtain the meaning of that jik?? number.

| T??ala number | Meaning | Classification | Sign | Quality |
|---|---|---|---|---|
| ı | Weakness | Weakening | | Neutral |
| ± | Omen | Heavy | ± | Negative |
| ¨ | Strength | Mild |  | Positive |
| © | Balance | Mild |  | Positive |
| © | Innumerability | Power |  | Positive |
| © | Multiple (excessive strength) | Heavy |  | Positive |
| © | Power | Power |  | Positive |





| | | | | |
|---|---|---|---|---|
| © | Multiple (excessive balance) | Heavy |  | Negative |
| © | Indomitability | Power |  | Positive |
| I | Mystery | Heavy |  | Negative |
| ıI | Grace | Mild |  | Positive |

The rules of permutation to derive the meanings are as follows:

1. The weakening number is a neutral number, and can be used to effect any number to change classification or quality or both
2. When the weakening number effects itself it changes quality and classification.
3. Any jik?? number is to be added up and effected to derive its meaning e.g. ©©©© = ±© (28). Here ±+ © = ıI and the meaning of ıI is grace.
4. Any jik?? number that when added exceeds one hundred is to be divided by the first number; if there are remainder, such should be added to the result. This process is to be continued until the result falls below one hundred and can yield a t??ala number.
5. No jik?? number whether below one hundred or not is to be effected unless it has been first added up.
   To effect means to permute the meaning of a number with another. Let us demonstrate a good example with ±©, ©© ©, ±©© (28,948,276). We add up this jik?? number to get:© © (46). Next we effect © to ©. This yields ıI (10) in the t??ala number series. And then we look up the meaning of ten which is grace.
6. The process of adding up jik?? number below one hundred is to be continued until it forms one of the t??ala numbers. The result is to be further added up until it forms a t??ala number before it is effected.

Hence, the Igbo numeric system clearly shows that numbers can have fixed meanings. This is novel but then it comes with some problems. It is understandable that number ± which means omen i.e. ± = ± É O. Thus, we may know in Igbo numeric epistemology that numbers have fixed meaning but how can we transfer this into the Arabic numeric system with which we conduct our mathematical enterprise? Let F be a numeric system; let G be the eaningfulness and let X be the free variable.

{X:Fx} = {X:Gx}É"x (Fx Gx)

That x is a formula of a numeric system equals x is meaningful, and this implies that for all x, x is a formula of a numeric system is equivalent to x is meaningful. But here comes Gödel's G2: if the above logical system is complete so as to prove Gx then it is not consistent which flaws the whole process; if on the other hand it is consistent, then Gx cannot be proved within it. However, if we assume that this second order logical system is consistent, it means we cannot prove Gx (i.e meaning of numbers) in it. This is the problem with the formal definition of meaning as referential. Here we have a logical system which suggests that numbers refer to their meanings, yet the consistency of the system shows that the referential attribute cannot be proved within it. Our system above shows:

(Fx = Gx) É" x (Fx Gx)

In a first order system this means that:"x (Fx © Gx) i.e. that Gx is provable or derivable in Fx. If this is the case then it is not the case that Gx is not provable or derivable in Fx i.e. x ~ (Fx © Gx); from which we derive the implication: "x (Fx © Gx) É x ~ (Fx © ~Gx). If this is granted then it is not the case that not Gx is derivable from Fx, i.e. "x (~Fx © ~ Gx). But this is what the G2 (Gödel's second incompleteness theorem proves of our second and first order formulas above. Yet the problem remains; how can we prove the consistency of a system that helps us derive Gx from Fx? This becomes a decision problem or as David Hilbert termed it the Entscheidungsproblem. And going by the results obtained by Alonzo Church (345 – 363: A Note, 40 – 41) Emil Post (103 – 105) and Alan Turing (230-265), there is no general algorithm for determining whether a given formula U of the functional calculus Z is provable, i.e. that there is no Turing machine (M) which given sufficient time and space, will eventually halt on input n.

Do we then take it to mean that our numbers have meanings only in the epistemology of natural language but cannot be proved in a formal language? Or, that the decision problem is soluble only that we have not





pursued its solution rigorously enough? Which ever one it is, philosophers of mathematics should never raise their heads with pride until they have discovered the meanings of the basic building blocks of the world's most exact science, or at least establish that numbers or mathematical objects have no fixed meaning in which case they will be meaningless for they will then, serve the mathematician as mere, empty symbols.

4. The Question of Mathematical Meaning

Philosophy of mathematics is conceived to investigate questions pertaining to the existence of mathematical objects, nature of mathematical propositions, nature of mathematical truth and the meaning of mathematical objects. Only recently, the question of mathematical understanding has been raised. Any sincere attempt to address the later question readily raises the question of mathematical meaning? The primitive of this question is the old inquiry: are numbers meaningful? To this I shall like to add the following collections: mathematics is a formal science, as a science of forms, does it have semantic imports or are the forms empty of meaning? Better still, is it the forms or the forms of things that are meaningful? Axioms or postulates are essential ingredients in mathematical proof; are they meaningful constructs in themselves, in the mind of the mathematician or are their meanings simply functional? Proof or provability is a basic character of the mathematical science; does the outcome of this exercise (proof) have any semantic import? In other words, do the results of proven formulas or theorems have any ontological meanings besides the formal adherence to the rules of such proofs? Are mathematical proofs empirical or rational? What is the relationship between mathematical entities and the objective world? What is the nature of mathematical meaning? Is it mental, physical, conventional, nominal, structural or functional? An attempt to supply answers to these questions of mathematical meaning would take us back to the foundations of mathematics. For this I shall have to go to the basics and define number.

Contrary to the views of Leibniz, Mill, Locke, Berkeley, etc., whose definitions or descriptions allocate content to number, I think number is simply a place-holder without contents, a fractional existent whose contents are things other than itself. This I think is what Frege wishes to say when he says number is objective but non-sensible. My position tallies with that of the logicist in that number is not objective in the sense of every other sensible thing. It differs from the logicist in that it is without content; its existence is fractional and dependent on empirical things. It is an empty symbol which serves as a sign of numbering, measurement and classification of empirical things. As a symbol therefore, it is empty of contents but a function of empirical entities. In this way it is like an empty can into which milk, food, beverages of assorted types or any sensible things can be stored. This is why it serves as a sign of numbering, measurement and classification of sensible things. Without number and by extension all mathematical objects, things in the world can hardly be classified, measured or grouped, therefore, it exists even though its existence is chiefly functional and dependent on the existence of empirical things. Number therefore cannot be existing as thought like the intuitionists say; nor can it exist outside of space and time as the Platonists contend; nor exist as mere ideas in the mind as the psychologists maintain; nor not exist as the norminalists and the conventionalists partly insinuate; nor is it an imaginary creation of the human mind as the fictionalist say. If anything, it is not like the formalist maintains an entity whose existence can be justified by ink and paper. If this is taken serious one might as well argue that the formalist is a kind of philosophical nominalist denying the existence of mathematical objects. If two things are said to stand in a certain numeric relation, do we mean that they do so in relation to some marks on paper? This is a subtle weakness in the formalist thought. Hence, if number were any one of these suppositions, then it would be nonsensical to apply it to sensible things.

Contrary to Hankel who states that number is indefinable and Leibniz who sees it as an adequate idea; I wish to say that number is definable and that it is not an idea much less an adequate one. It is an empty symbol which serves as a place-holder for classifying, measuring and grouping of sensible things. Ontologically, this makes number and indeed all mathematical objects, abstract and whose objective nature is dependent on the objective nature of sensible things. So my views differ with that of the logicist in that number is not only objective but objective in a pseudo sense. Semantically, it is my view that mathematical proposition can only be said to be true or false following its proof. In this way, the position of this paper tallies with that of the anti-realist who objects to the realist claim that mathematical propositions are either true





or false necessarily but differs with the anti-realist in that truth of propositions must be functionally derived. For us the functionalists, the truth of propositions are functionally derived when the meaning of such propositions reflects the facts in the world. If the realist view is correct, what is the place and need for proof? The intuitionist, Platonist, fictionalist and the conventionalist would not need proof for any of their formulas. Number therefore, is functionally meaningfully as a symbol and as an instrumentality of grouping, classifying and measuring sensible things. And so to the question what is the nature of mathematical meaning? I say it is functional- mathematical objects perform functions as place-holders for empirical entities and derive their pseudo-objective existence there-from. We see that one thing common among the formalist, intuitionist and logicist foundations of mathematics and the only one not denied, is the proof constructability of their results. This represents for us the functionalists, an indubitable frontline and an acceptance of the functionalist nature of mathematical meaning and by extension, functionalist foundations of mathematics.

The nature of mathematical meaning cannot be physical (physicalism) because mathematical objects are not essentially empirical entities; it cannot be mental (mentalism) because they are not essentially mental or properties of the intellect; it cannot be structural (structuralism) because mathematical objects do not constitute meaningless symbols justifiable only on paper, neither do proofs constitute empty games; it cannot also be nominal (nominalism) because mathematical objects are viable and active instruments of proof and therefore not insignificant or possibly non-existent; and finally; the nature of mathematical meaning cannot be conventional else how do we account for so many theories once accepted as true but later rejected as false? Mathematical objects are not meaningful by our convenient decision or values we assign to them by convention but simply by their peculiar functions.

Also, mathematical propositions as Frege said is analytic. It is impossible to construct a mathematical proposition which is axiomatizable whose subject term is not contained in its predicate term. If mathematics is not a science of proof, this probably would be possible, but since it is, it is impossible or so it seems at least! Hence, mathematical meaning also seeks beyond proof, the semantic content of proven results.

Hence, this paper rejects Frege's treatment of the meaning of mathematical objects. His context principle and his sense and reference have been shown in the body of this work to offer a nonsensical account of the meaning of mathematical objects. It is only when we as shown in this work discover thus far, that mathematical objects are only functionally meaningful that we would be able to grasp the worries of mathematical understanding: the reasons why propositions that are proved true or false actually hold. A good mathematical proof therefore is able to tell us that true and false propositions hold because the mathematical objects through their instrumental functions of grouping, classifying and measuring sensible things are able to give ontological commitment to such mathematical propositions such that their meaning confirm the facts they refer to. It is therefore only when we conceive number as wholly objective like the empiricists or with content like the logicist and the rest do, that we face the problem of mathsemantics as stated in the beginning of this work i.e. a lack of proof whether number is meaningful or meaningless.

## 5. Conclusion

We have in this work attempted to derive or discover the meaning of number having raised a problem to that effect which we christened mathsemantics. We showed that this problem is insoluble in so-far-as we allocate content to mathematical objects. In 2 answers to the question raised were sought in the views of famed authors across the tapestry of the history of our subject with no satisfying result. In 3 where we looked at the Igbo numeric system, we bumped into what satisfies the referential theory of meaning, in that numbers were seen to have fixed meaning but the problem arose when we tried to prove or derived the meaning of number from a formal or mathematical language. This landed our investigation in a new problem namely: the decision problem, which so far as philosophers of mathematics are concerned has no positive solution as yet.

What this shows therefore is, either the meanings of numbers as the Igbo numeric system helped us to discover cannot be proved in a formal system or that we have not worked hard enough to achieve this. But then, the Igbo numeric system has yet another problem: the whole of reality is so massive as to be represented by only eleven attributes corresponding to the eleven numbers which the Igbo say exist. If mathematics which has number as its basic tool truly accounts for many facets of reality, then reducing the massive structure of number to eleven





attributes comes short of fully accounting for the meaning of each individual number; not even the compositionality principle can possibly justify this. Despite this the Igbo numeric system remains the closest call to the goal of our investigation.

Again, if 2 + 2 is 4, and if 2 has fixed meaning then whenever we double it we should necessarily get 4 if and only if, 4 contains two 2s. But if numbers have tentative sense as Frege claims, then we can always add two 2s to get any number fitting for the context. This will mean that 2 + 2 may not necessarily be 4, and which is worse, if one thousand people performed this sum, it would be more accurate if one thousand different results are obtained. This is because; the sense of the number 2 will naturally vary from individual to individual.

On another hand, if actually numbers have fixed meaning then we cannot possibly have two of any number and it will be impossible to add two 2s to get 4 just the same way we cannot add two eagles to get eagle. This will make it easy for us to reduce the whole of number system to just eleven as the Igbo numeric system suggests or even fewer and pin down their individual meanings. But whereas this will make our investigation conclusive, it will on the other hand turn the discipline of mathematics into an absurd enterprise for then we might be unable to add, divide, multiple, subtract and indeed perform all the intricate exercises as we do in mathematics today. In 4 however, we articulated a theory of mathematical meaning, supplying a finite answer to the problem of mathsemantics. To actualize this we conceived and defined number differently- as a pseudo-objective and an abstract entity without content whose functions order and shape reality. We therefore, showed that the problem of mathsemantics is eliminated following this procedure.

There are therefore, four main deductions of our investigation namely: that numbers have tentative senses in that what Frege's context principle and the theory of sense and reference were able to show is that number takes different meanings according to context, and context determines sense of use; that the consistency of the result of Igbo numeric system which shows that numbers have fixed meanings cannot be proved in a first-order formal system; that the union of syntax and semantics of number created what we call the problem of mathsemantics which led to the failed set following an attempt to formalize it. The failed set tells us that given a non-empty set of empty sets, we would not be able to tell whether it is empty or not. When shifted to our inquiry concerning the meaning of number, it tells us that given any number whatsoever, we would not be able to tell whether it is an empty symbol or not, meaningful or not. Yes, number according to Frege has senses which vary from context to context, but that is another way of saying it has no own meaning. Hence, it is the position of this essay that whether number has meaning (denotative) or not, cannot out rightly be proved or disproved unless it is conceived in a pseudo-objective sense. Fourthly and finally, it is our position that the above three deductions are inadequate thereby necessitating our investigation in this paper. It is also our position here that number is not meaningless; that the tentative senses are not their meaning; that all mathematical objects are abstract and pseudo-objective in nature which we designate in this paper as functional and that their meaningfulness and truth-values can be adequately accounted for only in a functional sense. From this therefore, the question that justifies our endeavor in this work is: "if we could not as little as grasp the meaning of our mathematical objects, how could we reasonably talk of mathematical understanding?

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